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Existence, uniqueness and monotonic behavior of the solution of classical flow distribution problem for hydraulic networks with pressure-dependent closure relations

Leonid Korelstein[1,*]

[1]PSRE Co, Moscow, Russia

Abstract. Existence, uniqueness and monotonic behavior of the solution of classical flow distribution problem for hydraulic networks with pressure-dependent closure relations was proved. The closure relation can have very general form, restricted only by continuity and monotonicity conditions necessary for providing existence, uniqueness and continuity of flow distribution problem for each branch. It is shown that network as a whole "inherits" monotonicity and continuity of its branches behavior, and this provides existence and uniqueness of solution.

The question of existence and uniqueness of flow distribution always was in focus of interest of piping networks researchers ([1], [2], [3]). This problem was studied by Merenkov and Hasilev [4], and Suharev [5, 6]. The latest advanced research on the matter were done by Epifanov and Zorkaltsev, who prove existence and uniquesness of solution for the most general condition on edge closure relations for both classical flow distribution problem [7], and some non-classical variants [8, 9, 10, 11]. In all these works the problem was reduced to optimization of strictly convex function.

Nevertheless all mentioned results were obtained for pressure-independent edge closure relations. Such assumption is quite reasonable for isothermal incompressible liquid flow, but not valid in many cases of real gas flow and especially for gas-liquid flow, fluid properties highly depend on pressure. In this case there are no evident way to effectively reduce the problem to optimization task [6]. But (as a rule) it is possible to use modified variants of classical algorithms for finding solutions for such networks можно эффективно (such modified algorithms proposed by Mikhailovsky and Novitsky [12, 13, 14]). But lack of mathematically accurate proof of uniqueness and existence of solution remained some filling of dissatisfaction – especially because it is known for long time, that in similar problems of non-isothermal flow, especially gas-liquid flow, solution can be not unique!

It found out that for this kind of networks great role is played by properties of monotonicity (they were studies in latest researches of Chertkov – see for example [15]). Using them, the author managed to prove existence and uniqueness of solution of classical

*Corresponding author: korelstein@truboprovod.ru

flow distribution problem for very general conditions on pressure-dependent edge closure relations, which are direct generalization of Epifanov and Zorkaltsev results [7-11].

Already after writing the first variant of the article the author found out that relation between monotonicity of network flow distribution solution, and existence and uniqueness of the solution was discovered almost 60 years ago first by famous hydraulic network researcher Birkghoff and his colleague Kellogg [16], and then in more general form by famous mathematician Rheinboldt [17]. The later established it for the most general case of «abstract» network in frame of his study of multi-dimensional mappings, correspondent to different matrix types. It is very strange, but these results found out to be not noticed by hydraulic network specialists (except articles [18, 19] of that time) and unjust forgotten. The author hopes that this article will help to restore historical justice and these ideas will became natural part of hydraulic network theory.

1 Problem definition and closure relations

Let G - directed graph with N_V nodes (node set V) and N_E edges (edge set E). Flow rate X_i on edge i is related with start and end pressures P_{Fi} and P_{Li} by closure relation

$$X_i = \varphi_i(P_{Fi}, P_{Li}) \quad (1)$$

Let A – incidence matrix of graph G; Q - node inflow vector. Balance equations are

$$\mathrm{AX} = \mathrm{Q} \quad (2)$$

Using matrixes A_F and A_L, correspondent to starting and ending edges ($A = A_F + A_L$), vector P and vector Φ of functions φ_i, (1) can be written as

$$\mathrm{X} = \Phi(\mathrm{P_F}, \mathrm{P_L}),\ \ \mathrm{P_F} = \mathrm{A_F^T P},\ \ \mathrm{P_L} = -\mathrm{A_L^T P} \quad (3)$$

So we have $N_V + N_E$ equations for $2N_V + N_E$ unknowns (P, Q and X). But equations (2) are not independent – for connected graph G matrix A has rank $N_V - 1$, with additional equation:

$$\sum_{\mathrm{i}=1}^{\mathrm{Nv}} \mathrm{Q_i}=0 \quad (4)$$

So for connected graph G number of unknown is exceed by N_V number of independent equations, so values of N_V unknowns have to be set.

In classical flow distribution problem (CFDP) node pressure P_{fix} is set in $N_P > 0$ nodes (set V_P) and inflow Q_{fix} is set in remaining $N_Q = N_V - N_P$ nodes (set V_Q); pressures P_{var} are needed to be find in remaining N_Q узлах, plus flow rates X and inflows Q_{var} in N_P nodes with set pressure. The latest can be defined using equations (1) and (2), so it is enough to find P_{var}.

For «traditional» hydraulic networks functions φ_i depend only on pressure difference: $X_i = \varphi_i(P_{Fi} - P_{Li})$. Conditions were defined in [7] for functions φ_i (or invert functions f_i), which provide existence and uniqueness of CFDP solution for «traditional» hydraulic networks (Conditions A): 1) Continuity; 2) Strict monotonic increase; 3) Definition on all $\mathbb{R}$; 4) $\varphi_i(y) \to +\infty$ when $y \to +\infty$ and $\varphi_i(y) \to -\infty$ when $y \to -\infty$.

Condition 2 is necessary for uniqueness of solution; condition 1 and 2 are very natural and often fits in engineering practice. While conditions 3) and 4) are convenient mathematical extrapolation, which provides to guarantee existence of solution for all set values of pressures and inflows – in practice region of possible pressure is restricted – almost always from below (as a minimum, pressures should be positive), but also often by upper limits (by technological conditions, strength demands etc), so possible flow rates and inflows values are also restricted. So (as noted in [10]) in practice it would be good to have estimations of limit on set pressures and inflows, for which CFDP has solution in frame of value restriction on arguments of functions φ_i and f_i. But this is a separate task.

Set of functions, which fit conditions 1)-4), is noted as $\tilde{Z}_a$. It differs from introduced in [7-10] set $\tilde{Z}$ only by lack of conditions of zero pressure losses on zero flow rate. Set $\tilde{Z}_a$ can be created from $\tilde{Z}$ by addition of arbitrary constant. Functions from $\tilde{Z}$ correspond to passive edges, while functions from $\tilde{Z}_a$ also can represent active edges (with level difference, pumps, compressors etc).

Now let define set $\tilde{Z}_a^2$ of allowable pressure-dependent closure relations. It is natural generalization of $\tilde{Z}_a$. Function $\varphi \in \tilde{Z}_a^2$, if (Conditions A2): 1) $\varphi(P_F, P_L)$ is a continuous function of both variables P_F, P_L; 2) $\varphi(P_F, P_L)$ strictly decreases on P_L for any P_F; 3) $\varphi(P_F, P_L)$ strictly increases on P_F for any P_L; 4) $\varphi(P_F, P_L)$ is defined on all $\mathbb{R}^2$; 5) For any P_F $\varphi(P_F, P_L) \to -\infty$ when $P_L \to +\infty$, $\varphi(P_F, P_L) \to +\infty$ when $P_L \to -\infty$; 6) For any P_L $\varphi(P_F, P_L) \to +\infty$ when $P_F \to +\infty$, $\varphi(P_F, P_L) \to -\infty$ when $P_F \to -\infty$.

So set $\tilde{Z}_a^2$ consists of continuous function, which for any $P_L \in \tilde{Z}_a$ as functions of P_F and for any $P_F \in -\tilde{Z}_a$ as function of P_L. Examples of such functions are given in [12-14].

Like in case of $\tilde{Z}_a$, conditions 1)-3) are physically natural – while conditions 4)-6) are mathematical extrapolation.

Subset $\tilde{Z}^2$ of set $\tilde{Z}_a^2$ are functions, for which $\varphi(P, P) = 0$ for any P. Functions from $\tilde{Z}^2$ represent passive edges, while $\tilde{Z}_a^2$ covers also active ones.

Note that any pressure-independent closure relation which fits some condition from list A, also fits correspondent condition(s) from list A2. So all further results for pressure-dependent networks also are valid for "traditional" networks.

Note also, that introduction of function $\varphi(P_F, P_L)$, which gives unique value of flow rate on edge for pair of start and end pressure, is already some assumption. There are (rear) situations, when this is not right in practice – for example for centrifugal pumps with non-monotonic H-Q curve. However flow in this case can be unstable (so called pumping pompage).

2 Auxiliary definitions and properties

We will need to establish so properties of functions from $\tilde{Z}_a^2$.

Lets $\varphi \in \tilde{Z}_a^2$. The equations $\varphi(P_F, P) - X = 0$ and $\varphi(P, P_L) - X = 0$ on P define some implicit functions $f_L(P_F, \mathrm{X})$ and $f_F(P_L, \mathrm{X})$, which calculate end pressure by start pressure and flow rate, and vice versa. By definition of $\tilde{Z}_a^2$ these functions are defined on all $\mathbb{R} \times \mathbb{R}$, monotonic on both arguments, and go to infinity when any argument goes to infinity. These functions also are continuous on **both** arguments. This follows from strict monotonicity of functions $\varphi(P_F, P) - X$ and $\varphi(P, P_L) - X$ of P, and special «non-differential» variant of implicit function theorem ([20], [21]).

Note also, that direct of any edge is important only to correctly write balance equations (to assign correct sign to flow rate). The direction of the edge always can be inverted by replacing closure relation of inverted edge with $\varphi^*(P_F, P_L) = -\varphi(P_L, P_F)$.

Let G – connected graph, on which CFDP is defined. Let define the following operation (P-reducing, or $Pcut$) – consider all nodes with set pressure, and split all of them with degree >1 (Fig.1). As a result we will get graph $Pcut(G)$, in which all nodes with set pressure have degree 1 (see example on fig.2). Graph $Pcut(G)$ cen be non-connected, let G_{iP} – its connected subgraphs, and G_i – sungraphs, from which they are produced (see example on Fig.2). We will call G_i – P – components of graph G, and G_{iP} – its P-reduced components. P-components of graph intersect with each other only by nodes with set pressure, which connect them. We will call graph G^* P-reduced, if it is connected and $Pcut(G^*) = G^*$.

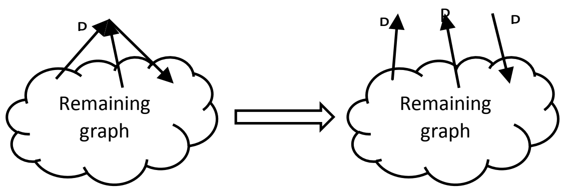

Fig. 1

It is evident, that to solve CFDP on G is the same as to solve it on each of G_i or G_{iP}. Source data of CFDP influences on solution only locally – i.e. change of set inflow or pressure has influence only in P-components, containing specific node with set pressure or inflow. For node with set inflow this is the only P-component, node with set pressure can be part of several P-components. In other words, set pressure nodes split graph on independent parts.

All P-reduced graphs have special structure (simplifying analysis) – they are separate edges with set node pressures, or connected subgraph with nodes with set inflows and edges which connect them, plus connected degree 1 nodes with set pressure. P-components of graph have the same structure, the only difference is that nodes with set pressure can have degree >1.

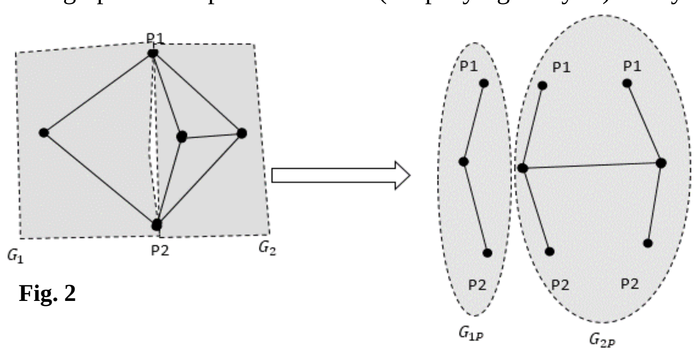

Fig. 2

Taking all said above into account, it is convenient to study properties of CFDP solution for P-reduced graphs – and then results usually can be easily generalized to any connected graph.

3 Uniqueness and monotonicity of CFDP

Let $\Omega \subseteq \mathbb{R}$ - some non-empty open connected set, and all functions φ_i are defined on $\Omega \times \Omega$. Ω can be $\mathbb{R}$, or infinite open interval, or open interval (restricted by low and upper pressure boundaries). Let us first consider function φ_i, which are strictly monotonic, but not necessarily continuous. It found out that this is already enough to guarantee uniqueness and monotonicity of CFDP solution.

Let $P \in \Omega^{N_V}$ – node pressures vector. Then equations (1) and (2) completely define flow rates and nodal inflows. We will study how nodal pressures change influence on nodal inflows.

Lemma 1 (about nodal inflows).

Let G – connected graph with strictly monotonic edge closure relations, P^1 and P^2 – nodal pressure vectors, X^1, X^2, Q^1, Q^2 – flow rate and inflow vectors defined by (1) and (2). Let's consider the following sets of nodes: 1) V^+ - set of nodes, for which $P_i^2 > P_i^1$ (pressure increased); 2) V^- - set of nodes, for which $P_i^2 < P_i^1$ (pressure decreased); 3) V^0 - set of nodes, for which $P_i^2 = P_i^1$ (pressure didn't change); 4) V^{0+} - subset of V^0 nodes, connected with any node from V^+, but not connected with any nodes from V^-; 5) V^{0-} - subset of V^0 nodes, connected with any node from V^-, but not connected with any nodes from V^+; 6) $V^{0\pm}$ - subset of V^0 nodes, connected with both nodes from V^+ and V^-; 7) V^{00} - subset of V^0 nodes, not connected with any node from V^+ or V^-.

Then: 1) For any node $v \in V^{00}$ $Q^2(v) = Q^1(v)$ (inflow doesn't change); 2) For any node $v \in V^{0+}$ $Q^2(v) < Q^1(v)$ (inflow decreases); 3) For any node $v \in V^{0-}$ $Q^2(v) > Q^1(v)$ (inflow increases); 4) If $V^+ \neq \emptyset$ and $V^+ \neq V$, then $\sum_{v\in V^+} Q^2(v) > \sum_{v\in V^+} Q^1(v)$; 5) If $V^- \neq \emptyset$ and $V^- \neq V$, then $\sum_{v\in V^-} Q^2(v) < \sum_{v\in V^-} Q^1(v)$.

Note, that it is impossible to say a priory anything about inflows in $V^{0\pm}$ nodes.

Proof of Lemma 1.

Let v – graph node. For simplification we can change directions of connected edges so that they all start in v. According equation (1) inflow $Q(v)$ in v is a sum of all flow rates in connected edges.

For $v \in V^{00}$, for all edges connected with v end nodal pressures P_i^2 are equal P_i^1, so all flow rates also are equal, and $Q^2(v) = Q^1(v)$.

If $v \in V^{0+}$, for all edges started in v start pressure doesn't change ($P^2(v) = P^1(v)$), while end pressure in end node v_L does not change ($P^2(v_L) = P^1(v_L)$) or increases ($P^2(v_L) > P^1(v_L)$). In first case flow rate doesn't change, in second case it decreases (as edge closure relation is monotonic). As $v \in V^{0+}$, there exist at least one edge with decreased flow rate. So $Q^2(v) < Q^1(v)$, which prove 2nd item of lemma. 3rd item proof is similar.

Let $V^+ \neq \emptyset$ and $V^+ \neq V$. Let us sum all equations (1) for nodes from V^+. In resulting sum flow rates on edges connected nodes from V^+, will nullify each other. As $V^+ \neq V$ and graph is connected, set of edges, connected nodes from V^+ with other nodes (from $V \setminus V^+$), is not empty. For simplicity let's change their direction if necessary, so all start nodes were from V^+, and end ones from $V \setminus V^+$. The sum of all inflows on all V^+ nodes is equal to sum of flow rates on edges from V^+ «outside»: $\sum_{v\in V^+} Q(v) = \sum_{v\in V^+, u\in V\setminus V^+} X(e_{vu})$. On each of such edges transition from P^1 to P^2 increase start node pressure, while end node pressure decreases or does not change. As close relation are monotonic, this produces flow rate increase on each of the edges, and increase of flow rate sum – and this proves item 4 of the lemma. Item 5 can be proved in a similar way.

Now we will find out CFDP solution properties which follow from closure relations monotonicity.

Theorem 1 (Uniqueness of CFDP solution).

Let G – connected graph with strictly monotonic edge closure relations. Then CFDP solution for G is unique.

Proof of Theorem 1.

In special case, when $N_P = N_V$, theorem is obvious.

Let's consider case $0 < N_P < N_V$, (i.e. $V_P \neq \emptyset$ и $V_Q \neq \emptyset$). Let's suppose that for the same sets V_P and V_Q, and the same nodal pressures and inflows there exist 2 different solutions of CFDP (noted by indexes 1 and 2). Then this solution should have different vectors of nodal pressures. Let's apply lemma 1. As in V_P nodes pressures are set, $V^+ \subseteq V_Q$, i.e. $V^+ \neq V$. Similar $V^- \subseteq V_Q$, and $V^- \neq V$. If $V^+ \neq \emptyset$, item 4 of lemma 1 gives $\sum_{v\in V^+} Q^2(v) >$

$\sum_{v \in V^+} Q^1(v)$. But this is impossible, as $V^+ \subseteq V_Q$, and all inflows in V_Q are set and don't change. So $V^+ = \emptyset$. Similar way we can get $V^- = \emptyset$. But this means, that $V^0 = V$, i.e. all nodal pressure of solutions 1 and 2 are equal, so all flow rates and inflows are also equal. This proves theorem 1.

Now let's study how CFDP solution depends on change of set nodal pressures and inflows The following theorem generalizes properties established in [15] for "classical" network, to the networks with pressure-dependent closure relations.

Theorem 2 (Monotonicity of CFDP solution).

Let G – connected graph with strictly monotonic edge closure relations, with 2 CFDP (problems 1 and 2) defined with the same non-empty sets V_P and V_Q, and set nodal pressures $P_{fix}^{(1)}$ and $P_{fix}^{(2)}$, and set inflows $Q_{fix}^{(1)}$ and $Q_{fix}^{(2)}$, related as $P_{fix}^{(1)} \le P_{fix}^{(2)}$ и $Q_{fix}^{(1)} \le Q_{fix}^{(2)}$, and solutions of both problems exist . Let V_P^+ and V_Q^+ - node sets with set nodal pressures and inflows, for which inequalities are strict. Then: 1) For all P-components of G, where are nodes from V_P^+ or V_Q^+, $\forall v \in V_Q$ $P^2(v) > P^1(v)$. In other P-components $\forall v \in V_Q$ $P^2(v) = P^1(v)$; 2) If $N_P = 1$ and $V_Q^+ \neq \emptyset$, then in the only node v with set pressure $Q_{var}^{(1)}(v) > Q_{var}^{(2)}(v)$. Otherwise $Q_{var}^{(1)}(v) = Q_{var}^{(2)}(v)$; 3) If $N_P > 1$, then $Q_{var}^{(1)}(v) > Q_{var}^{(2)}(v)$ for $\forall v \in V_P \setminus V_P^+$ from any P-component, containing nodes from $V_Q^+ \cup V_P^+$. For other $\in V_P \setminus V_P^+$ $Q_{var}^{(1)}(v) = Q_{var}^{(2)}(v)$; 4) If $N_P > 1$, $V_Q^+ = \emptyset$, $V_P^+ \neq \emptyset$, $V_P \setminus V_P^+ \neq \emptyset$, then $\sum_{v \in V_P^+} Q_{var}^{(1)}(v) < \sum_{v \in V_P^+} Q_{var}^{(2)}(v)$.

Proof of Theorem 2.

It is enough to prove the theorem for each P-component. In those of them, where there are no nodes from V_P^+ or V_Q^+, CFDP data is the same, so according Theorem 1 the solution is the same – and this proves last parts of the theorem items 1 and 3.

Let G^* -P-component with nodes from V_P^+ or V_Q^+. We will apply Lemma 1 for it.

For G^* $V_P \neq \emptyset$ and $V_P \subseteq V^0 \cup V^+$, thus $V^- \neq V$. According item 5 of lemma 1 sum of inflows on all nodes from V^- must decrease. But this is impossible - V^- can include only nodes with set inflows, where inflows don't change! Thus $V^- = \emptyset$. So all nodes of G^* are contained in $V^0 \cup V^+$.

Suppose that $V_Q \cap V^0 \neq \emptyset$. Consider subgraph G^{**} of nodes V_Q, nodes V_P^+ and edges which connect them. As G^* is P-component, subgraph G^{**} is connected. Then $V_Q \cap V^{0+} \neq \emptyset$. But according lemma 1 in nodes $V_Q \cap V^{0+}$ inflows have to decrease – this contradicts the theorem conditions. Thus $V_Q \cap V^0 = \emptyset$ и $V_Q \subset V^+$, which proves 1st item of the theorem.

All above shows that $V^0 = V_P \setminus V_P^+$. Further, $V^{00} = \emptyset$ - otherwise nodes V^{00} would not be connected with other nodes of G^*. So $V_P \setminus V_P^+ = V^{0+}$, and according Lemma 1 inflows in these nodes decrease, and this proves items 2 and 3.

Item 4 evidently follows from item 3 and equation (4).

4 Continuity of CFDP solution

Now we will add continuity condition to strict monotonicity of functions φ_i and will study what additional properties of CFDP solution it brings.

Note by Y vector, contained with CFDP source data – first N_P components – set nodal pressures P_{fix}, other N_Q – set inflows Q_{fix}; $Y \in \Omega^{N_P} \times \mathbb{R}^{N_Q}$. Note by E set of Y, for which CFDP is solvable. What can we say about E?

Theorem 3 (Continuity and monotonicity of CFDP solution).

Let G – Connected graph with continuous and strictly monotonic edge closure relations. Then:

1. E is homeomorphic to Ω^{N_V} (or $\mathbb{R}^{N_V}$, what is the same) and thus non-empty, open and connected.
2. All solutions parameter (nodal pressures, flow rates, inflows) are continuous functions of source data.
3. Solution is monotonic on source data:
   a. Pressures in nodes from V_Q strictly increase when set inflows and nodal pressures increase in their P-component, and don't depend on source data in other P-components.
   b. Inflows in all nodes from V_P strictly decrease when set inflows increase in their P-components, and don't depend on set inflows in other P-components.
   c. Inflow in node from V_P:
      i. Strictly increases with set pressure increases in the same node, if there are other nodes in V_P.
      ii. Strictly decreases while set pressures in other nodes increase from the same P-component.
      iii. Does not change in all other cases.

<u>Proof of theorem 3.</u>

Consider mapping $\Psi\colon \Omega^{N_V} \to \Omega^{N_P} \times \mathbb{R}^{N_Q}$, which maps nodal pressure vector P to vector Y, which consists from pressures P_i in nodes from V_P and inflows Q_i in nodes from V_Q, calculated from P using equations (1)-(2). In fact mapping Ψ maps vector P to source data of CFDP for which it is a solution. So $E = \Psi(\Omega^{N_V})$.

As functions φ_i are continuous, mapping Ψ is also continuous. As functions φ_i are strictly monotonic, according theorem 1 this mapping is also injective. So according Brouwer invariance of domain theorem, mapping Ψ is homeomorphism, and its image $E = \Psi(\Omega^{N_V})$ is open and homeomorphic to Ω^{N_V}.

As Ψ is homeomorphism, invert mapping Ψ^{-1} is continuous on E. This means that nodal pressures continuously depend on source data of CFDP. As functions φ_i are also continuous, flow rates and inflows (calculated by equations (1), (2)) also depends continuously on CFDP source data.

3rd item of the theorem directly follows from theorem 2.

<u>Theorem 4 (CFDP solution existence for intermediate source data).</u>

Let G – connected graph with strictly monotonic and continuous edge closure relations, and there is CFDP on G with $N_Q > 0$.

If $Y^1 = \left(P^1_{fix}, Q^1_{fix}\right) \in E$, $Y^2 = \left(P^2_{fix}, Q^2_{fix}\right) \in E$, $P^1_{fix} \le P^2_{fix}$, $Q^1_{fix} \le Q^2_{fix}$, and for $\left(P_{fix}, Q_{fix}\right)$ $P^1_{fix} \le P_{fix} \le P^2_{fix}$ $Q^1_{fix} \le Q_{fix} \le Q^2_{fix}$, then $\left(P_{fix}, Q_{fix}\right) \in E$ and for CFDP solution $P^1_{var} \le P_{var} \le P^2_{var}$.

In other words, if CFDP has is solvable for some "boundary" pressures and inflows, it is solvable for intermediate source data.

<u>Proof of theorem 4.</u>

Let K - set $P^{(1)}_{fix} \le P_{fix} \le P^{(2)}_{fix}$, $Q^{(1)}_{fix} \le Q_{fix} \le Q^{(2)}_{fix}$ in $\Omega^{N_P} \times \mathbb{R}^{N_Q}$. In fact we need to prove, that $K \subseteq E$, i.e. $K \setminus E = \emptyset$**.** Suppose that $K \setminus E \neq \emptyset$. As K is compact, and E is open, $K \setminus E$ is also compact. So $K \setminus E$ contains some point Y^*, nearest to Y^1. Consider interval $I^* = [Y^1, Y^*]$, connecting points Y^*, Y^1. As Y^* is the nearest to Y^1 point in $K \setminus E$, all points of I^*, except Y^*, $\in E$. Consider mapping Ψ^{-1} on interval $[\,Y^1, Y^*)$. According theorem 4 it is continuous, increases monotonically on all coordinates and upper limited by coordinates of $\Psi^{-1}(Y^2)$. So $\Psi^{-1}(Y)$ has some limit P^*, when $Y \to Y^*$ on I^*. Ψ is continuous, so $\Psi(P^*) =$

Y^*, and $Y^* \in E$ – contradiction, which proves $K \setminus E = \emptyset$. Inequality $P_{var}^1 \le P_{var} \le P_{var}^2$ follows from item 3a of theorem 4.

5 CFDP solution existence theorem

Theorem 5 (Existence of CFDP solution).

Let G – connected graph with all edges closure relation from $\tilde{Z}_a^2$, for which CFDP is defined. Then solution of such CFDP always exists.

Proof of theorem 5.

We will prove the theorem by induction on number of edges N_E of graph G. In fact it will describe some recursive algorithm for finding the solution, which allows to establish its existence.

The base of induction ($N_E = 1$). For Graph with one edge (and two nodes), for the case, when pressure is set in all nodes, CFDP is solvable by definition. If pressure is set in one node and inflow in another one, CFDP solution exists, because finctions f_L and f_F for edges with closure relations from $\tilde{Z}_a^2$ are defined on all $\mathbb{R} \times \mathbb{R}$ (see section 2).

The step of induction. Let graph G has $N_E > 1$ edges and CSFP solution existence is proved for all graphs with edges number less than N_E. Consider graph $G' = Pcut(G)$. CSFP solution on graph G is the same as on G'. If the latest is non-connected and contains several connected subgraphs (P-reduced components), each of them contain < N_E edges, so on each of them CSFP has solution, and union of these solutions gives solution on G'.

The case remains when G' is connected and is P-reduced. Select some node v' with set pressure $P_{fix}(v')$ in G'. This node has degree 1 in the graph an is connected by edge e' with some node v'' with set inflows $Q_{fix}(v'')$. For further simplification, we can change direction of e', if necessary, so it starts in v' and ends in v''. Let G'' - graph, produced from G' by deleting node v' and edge e' (Fig.3). It is connected and has $N_E - 1$ edges, so on it any CFDP is solvable. There can be 2 cases – 1) G' has only one node with set pressure (node v'); 2) G' has more than one node with set pressure – so G'' also contains nodes with set pressure.

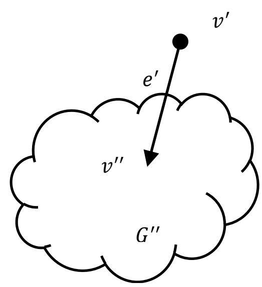

Fig. 3

1st case. In this case we can do «direct calculation» of edge e'. From equation (4) we need to have $Q(v') = -\sum_{v\in G', v\neq v'} Q_{fix}(v)$. As v' has degree 1, let take flow rate on e' equal to $X(e') = Q(v')$ and calculate pressure in v'' using f_L for e': $P(v'') = f_{e'L}\left(P_{fix}(v'), X(e')\right)$. Set CFDP on G'' with the same source data, as on G', but instead of inflow in v'' let set pressure in it. Solution of such CFDP on G'' exists, and along with set pressure in v' will provide solution of original CFDP on G'. This is evident for all nodes, except v''. In v'' for found solution $Q(v'') = -\sum_{v\in G'', v\neq v''} Q_{fix}(v) - X(e') = -\sum_{v\in G'', v\neq v''} Q_{fix}(v) + \sum_{v\in G', v\neq v'} Q_{fix}(v) = Q_{fix}(v'')$.

2nd case. In this case we will find such pressure P'' in v'', so CFDP on G'' with set pressure P'' in v'', and the same conditions in other nodes of G'', as on G', in combination with flow rate on e', correspondent to $P_{fix}(v')$ and P'', would provide solution of original CFDP on G'. To do this, we only need to have set inflow in v'', i.e. to have $Q_{G''\,var}(v'', P'') - \varphi_{e'}\left(P_{fix}(v'), P''\right) = Q_{fix}(v'')$, where $\varphi_{e'}$ is function φ for e', $Q_{G''\,var}(v'', P'')$ - inflow in v'' for solution of CFDP on G'' with set pressure P'' in v'' and the same condition in other nodes. Consider function $q(P'')$:

$$q(P'') = Q_{G''\,var}(v'', P'') - \varphi_{e'}\left(P_{fix}(v'), P''\right) - Q_{fix}(v'')$$

Function $-\varphi_{e'}\big(P_{fix}(v'), P''\big)$ of P'' is from set $\tilde{Z}_a$. $Q_{G''\,var}(v''\,, P'')$ as function of P'' is defined on all $\mathbb{R}$. According item 2 of theorem 3 it is continuous on P'', and according item 3ci of theorem 3 increases monotonically. Thus, function $q(P'')$ also $\in \tilde{Z}_a$, so should equal 0 in some point. This point is the value of P'', which we are looking for.

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