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On the Convergence of "Estimated Flow Rate" Method for Hydraulic Network Flow Rate Distribution Analysis

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Abstract. The article contains prove that Estimate Flow Rate (EFR) method of hydraulic network flow rate distribution analysis described in [1-3], converges from any initial point for some class of hydraulic networks with odd functions of losses on edges. Method modification was proposed which provide such convergence for more wide class of hydraulic networks.

The iterative method of "Estimated Flow Rate" (or EFR), described in article [1], was proposed by Duginov several decades ago and was used successfully from 1972 [2, 3] for hydraulic and other network analysis.

Various methods of solving classical hydraulic network flow rate distribution problem have been investigating already during almost a century, starting from Lobachev – Hardy Cross method [4, 5]. The most popular in the last period are different variants of iterative methods based on Newton-Raphson algorithm, which use derivative linearization of hydraulic network equations in different forms: methods of Linear Theory (LT) [8], Loop Flow (LP) [6, 7, 9, 10], Nodal Adjustment method (NA) [6, 7, 11, 12], and Global Gradient Algorithm (GGA) [13, 14]. The review of these methods can be found in [15, 16], where their uniform nature is shown. Another recent variant of such methods is Reformulated Co-Tree Method (RCTM) [17], combining GGA and LP and more effective in some cases (but demanding topological analysis of the network). Further efforts to make LP more effective can be found also in [18]. The speed of convergence of these methods is usually quite high, but convergence can significantly depend on initial point.

In the same time some "non-Newtonian" approaches were considered based on chord or secant network equations linearization. Such methods were originally proposed in [19, 20] and then in [7], and also mentioned in [21]. Systematic research of such methods was done in article [22]. Note that many of such methods found out to be very efficient.

EFR method also belongs to chord linearization type of methods and has linear speed of convergence, but this speed is quite good (especially for flow rates) – usually in practice no more than 10 iterations are required. The biggest advantage of the method is its high stability and practical independency of initial point. As many years of usage experience showed, EFR allows to set practically any initial flow rate and node pressure values, and number of iterations is almost independent of these values. This advantage of EFR allows to apply it

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also in combination with other methods (for example GGA) to get better initial approximation for them.

However so far there was no formal mathematical prove of EFR convergence, and no enough understanding of the nature of its stability. This article fills this gap. It found out that to do this EFR should be considered in frame of optimization problem.

1. Problem definition and EFR method

Let G - directed connected graph with N_V nodes (node set V) and N_E edges (edge set E).

Well known hydraulic network equations of classical flow rate distribution problem (CFDP) can be written as

$$A^T P = F(X) \tag{1}$$

$$AX = Q \tag{2}$$

Where A – incidence matrix of graph G, X – vector of mass flow rates, Q – node mass inflow vector, P – vector of node potentials (pressures), $F(X)$ – vector function, each element of which is function of losses on edges.

It is supposed that on node subset V_P node pressures P_{fix} are set (in $N_P > 0$ nodes), and in remaining N_Q nodes (subset V_Q) node inflows Q_{fix} are set, and the problem is to find such node pressures P_{var} in N_Q nodes (and correspondently inflows Q_{var} in N_P node and edge flow rates X), which satisfy equations (1), (2).

The idea of EFR is to replace on each iteration original equations (1), (2) by equations with $F(X)$ replaced by linear vector $F_{EFR}(X) = A_{EFR}X + F_0$, $F_0 = F(0)$, correspondent to chord of function $F(X)$ (or invert to F function φ), connected point of current iteration and point of zero flow rate.

So on i-th iteration the following system of linear equations is solved

$$A^T P^{(i)} = A_{EFR}^{(i)} X^{(i)} + F_0 \tag{3}$$

$$AX^{(i)} = Q \tag{4}$$

Here elements of diagonal matrix $A_{EFR}^{(i)}$ (which can be called edge linear resistances) on each iteration are calculated as

$$a_{EFRjj}^{(i)} = \left(y_j^{(i-1)} - F_{0j}\right)/X_{EFRj}^{(i-1)}, \tag{5}$$

where «estimated flow rate» $\mathrm{X}_{\mathrm{EFRj}}^{(i-1)} = \varphi\left(y_j^{(i-1)}\right)$, $Y^{(i-1)} = A^T P^{(i-1)}$.

In the seldom case when flow rate on the edge equals zero, and formula (5) is undefined, $a_{EFRjj}^{(i)}$ is set as $a_{EFRjj}^{(i)} = \frac{dF_j}{dX_j}(0)$, where derivative is calculated analytically or numerically (in order this derivative be non-zero, laminar type of the flow in this case should be considered [24]).

As F function usually is monotonic and smooth enough [24], effective calculation of inverse function on every iteration is not a problem, as there are now known many highly effective root finding algorithms for solving such problems.

Note that on initial step it is possible (and often more effective) to set some reasonable edge flow rates $\mathrm{X}_{\mathrm{EFR}}^{(0)}$ instead of node pressures, as in (5) we can set $Y^{(0)} = F\left(\mathrm{X}_{\mathrm{EFR}}^{(0)}\right)$.

Note also that on each iteration original network is replaced by linear one with similar edge properties – passive edges remain passive, and active edges have the same head at zero flow rate - so for solutions on each iteration the same theorems (proven in [25]) for network solutions in restricted node pressure space is applied, as for original network problem, and already after the first iteration node pressure values are usually close enough to the solution, and this minimizes risk of occasional «blowout» of node pressures during iterations into zone where the model is not adequate (fluid is boiling or condensing, or choked flow occurs). This is one of the reasons of EFR is so stable to initial point selection.

2. Convergence of EFR method

We will prove EFR method convergence of hydraulic networks with edge loss functions satisfying the following conditions:

1) $f_i \in C[-\infty, +\infty]$, i.e. defined and continuous on $\mathbb{R}$;
2) f_i strictly monotonically increases;
3) $f_i \to -\infty$ for $x_i \to -\infty$ and $f_i \to +\infty$ for $x_i \to +\infty$;
4) Functions $a_{EFRii}(x_i) = [f_i(x_i) - f_i(0)]/x_i$ monotonically increase for $x_i > 0$;
5) The derivative of f_i at $x_i = 0$ exists and positive: $f_i'(0) > 0$;
6) Functions $f_i(x_i) - f_i(0)$ are odd.

Conditions 1-3 have evident physical sense and coincide with conditions on edge functions, for which existence and uniqueness of CFDP solution were proved [23].

Condition 4 is weak variant of convex condition of f_i for $x_i > 0$ and practically always is satisfied. If f_i for $x_i > 0$ are convex, this condition is satisfied.

Condition 5 (at least in the part of left-hand and right-hand derivatives existence) also has clear physical sense – edge resistance in the region of laminar flow should be correctly taken into account – and always can be satisfied by adding corresponding linear part into edge loss function.

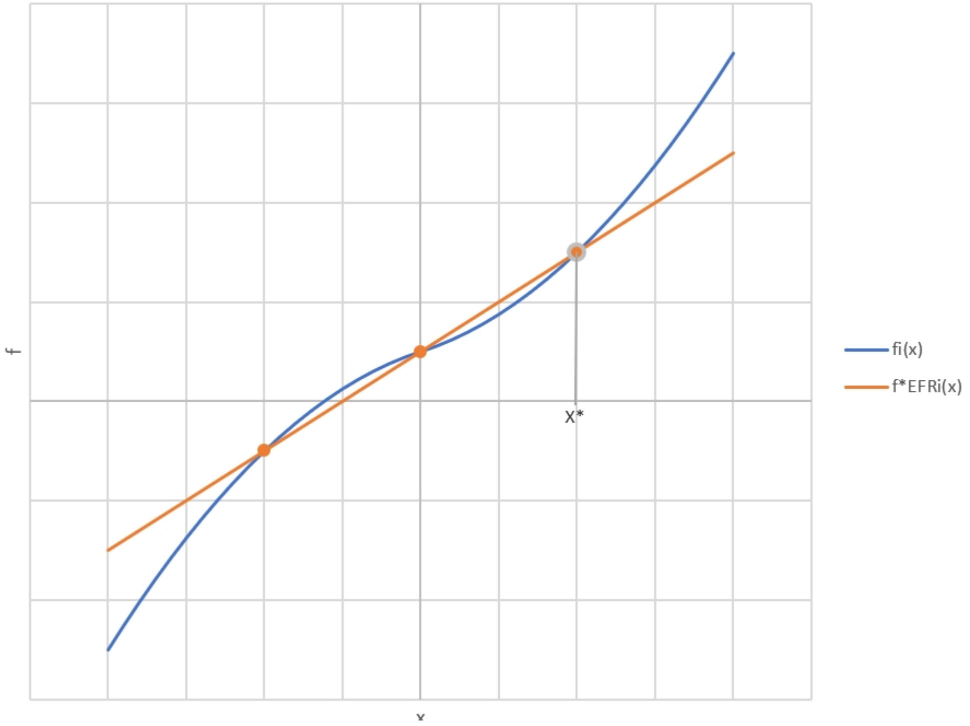

Fig. 1

In the same time condition 6 not always can be satisfied. It means in fact that behavior of edge is independent of flow direction. In most cases this is right – but if the edge includes reducers, pumps or other elements with different behavior for different directions of the flow, condition 6 for such edge will not be satisfied. In next section we will consider how EFR can be modified to drop this condition.

Note that conditions 1-6 imply that functions $a_{EFRii}(x_i)$ (also defined in 0 as $a_{EFRii}(0) = f_i'(0)$) are continuous on $\mathbb{R}$, even and increase monotonically for $x_i \geq 0$.

Also conditions 4-6 imply that for $\forall x^*$ and correspondent $f^*_{EFRi}(x_i)$ the following inequalities are valid (Fig.1)

$$f_i(x_i) \geq f_i(0) + a_{EFRii}(x^*)x_i = f^*_{EFRi}(x_i) \quad \text{for} \quad x_i \geq |x^*| \quad \text{and} \quad -|x^*| \leq x_i \leq 0 \tag{6}$$

$$f_i(x_i) \leq f_i(0) + a_{EFRii}(x^*)x_i = f^*_{EFRi}(x_i) \quad \textit{for} \quad x_i \leq -|x^*| \quad \textit{and} \quad 0 \leq x_i \leq |x^*| \tag{7}$$

For inverse functions $\varphi_i = f_i^{-1}$ and $\varphi^*_{EFRi} = f^{*-1}_{EFRi}$ the following correspondent inequalities are valid for $y^* = f_i(x^*)$

$$\varphi_i(y_i) \leq \varphi^*_{EFRi}(y_i) \;\; \textit{for} \;\; y_i - f_i(0) \geq |y^* - f_i(0)| \;\; \textit{and} \;\; -|y^* - f_i(0)| \leq y_i - f_i(0) \leq 0 \tag{8}$$

$$\varphi_i(y_i) \geq \varphi^*_{EFRi}(y_i) \quad \textit{for} \quad 0 \leq y_i - f_i(0) \leq |y^* - f_i(0)| \quad \textit{and} \quad y_i - f_i(0) \leq -|y^* - f_i(0)| \tag{9}$$

Let's consider mapping $M_{EFR}:\mathbb{R}^{N_Q} \to \mathbb{R}^{N_Q}$, which maps arbitrary node pressure vector P_{var} in nodes V_Q, to node vector obtained by solving equations (3), (4). The mapping M_{EFR} is continuous (as functions $a_{EFRii}(x_i)$, which are coefficients of nonsingular linear system (3), (4), are continuous) and has the only fixed point, which correspond to existing and unique original CFDP solution [23].

It is well known [7, 15, 23] that CFDP can be reformulated as the problem of minimization on $P_{var} \in \mathbb{R}^{N_Q}$ of function

$$\Phi(P_{var}) = \sum_{i=1}^{N_E} \int_0^{y_i} \varphi_i(u)du - \sum_{i=1}^{N_Q} Q_i y_i, \tag{10}$$

Where vector of pressure drops $Y = A^T P$, $\varphi_i = f_i^{-1}$ – invert functions to f_i.

Function $\Phi(P_{var})$ is continuous (and even continuously differentiable) and has one global minimum, which corresponds to CFDP solution. In the same time $\Phi(P_{var}) \to +\infty$ for $|P_{var}| \to +\infty$, so for any $\mathrm{C} > \Phi_{min}$ – minimal value of $\Phi(P_{var})$ – the set of those P_{var}, that $\Phi(P_{var}) \leq \mathrm{C}$, is not only closed, but bounded, and thus is compact set.

It is found out that mapping M_{EFR} has the following wonderful property – for any P_{var} not coinciding with its fixed point (i.e. CFDP solution) it strictly decreases the value of function Φ:

$$\Phi\big(M_{EFR}(P_{var})\big) < \Phi(P_{var}) \tag{11}$$

Let's take any vector P^*_{var}, then $M_{EFR}(P^*_{var})$ is the solution of modified CFDP (3), (4). Finding this solution is also equivalent to finding a minimum of function

$$\Phi^*_{EFR}(P_{var}) = \sum_{i=1}^{N_E} \int_0^{y_i} \varphi^*_{EFRi}(u)du - \sum_{i=1}^{N_Q} Q_i y_i, \tag{12}$$

The point of minimum doesn't change when we add constant to minimized function, so we can minimize function $\Phi^{*\prime}_{EFR}(P_{var}) = \Phi^*_{EFR}(P_{var}) + \Phi(P^*_{var}) - \Phi^*_{EFR}(P^*_{var})$, which coincide with $\Phi(P_{var})$ at $P_{var} = P^*_{var}$. Then it found out that this function majorizes $\Phi(P_{var})$, i.e. for $\forall P_{var}$

$$\Phi^{*\prime}_{EFR}(P_{var}) = \Phi^*_{EFR}(P_{var}) + \Phi(P^*_{var}) - \Phi^*_{EFR}(P^*_{var}) \geq \Phi(P_{var}) \tag{13}$$

Really, from (10) and (12)

$$\Phi^*_{EFR}(P_{var}) + \Phi(P^*_{var}) - \Phi^*_{EFR}(P^*_{var}) - \Phi(P_{var}) = \sum_{i=1}^{N_E} \int_{y_i^*}^{y_i} [\varphi^*_{EFRi}(u) - \varphi_i(u)]du \tag{14}$$

Let's consider terms in (14) – functions $g_i(y_i) = \int_{y_i^*}^{y_i} [\varphi^*_{EFRi}(u) - \varphi_i(u)]du$ (Fig.2).

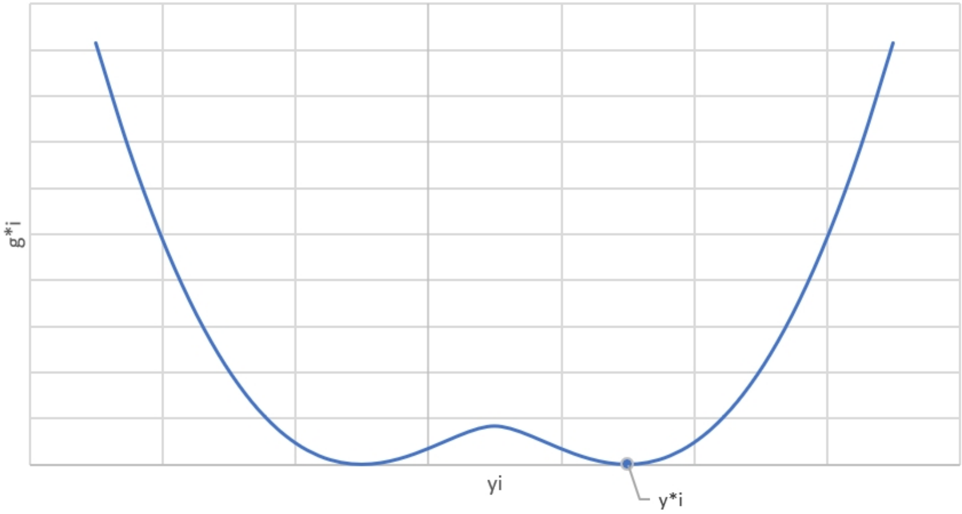

Fig. 2

We have $g_i(y_i^*) = 0$. When $y_i - f_i(0)$ has the same sign, as $y_i^* - f_i(0)$, $g_i(y_i)$ is non-negative according inequalities (8), (9). As $f_i(x_i) - f_i(0)$ is odd, function $g_i(y_i)$ is symmetrical regarding line $y_i = f_i(0)$, and so it is non-negative also when $y_i - f_i(0)$ and $y_i^* - f_i(0)$ has different signs. In particular

$$\int_{y_i^*}^{f_i(0)} [\varphi_{EFRi}^*(u) - \varphi_i(u)]du = \int_{2f_i(0)-y_i^*}^{f_i(0)} [\varphi_{EFRi}^*(u) - \varphi_i(u)]du \tag{15}$$

So $g_i(y_i) \geq 0$, and this proves (13).

As $M_{EFR}(P_{var}^*)$ is the only minimum of $\Phi_{EFR}^{*\prime}(P_{var})$, for $M_{EFR}(P_{var}^*) \neq P_{var}^*$ from (13) we get

$$\Phi\big(M_{EFR}(P_{var}^*)\big) \leq \Phi_{EFR}^{*'}\big(M_{EFR}(P_{var}^*)\big) < \Phi_{EFR}^{*'}(P_{var}^*) = \Phi(P_{var}^*) \tag{16}$$

This proves (11).

Established properties of M_{EFR} and function Φ allow to prove convergence of EFR method for any initial point! This follows from simple general topological theorem.

Theorem 1.

Let K – nonempty compact set, on which continuous function $\Phi: K \to \mathbb{R}$ is defined, which has the only global minimum in $u^* \in K$, and continuous mapping $M: K \to K$ is defined, such that $M(u^*) = u^*$ and for $\forall u \in K$, not equal to u^*, $\Phi\big(M(u)\big) < \Phi(u)$.

Then for any $u^{(0)} \in K$ sequence $u^{(i+1)} = M\big(u^{(i)}\big)$ converges to u^*, and this convergence is uniform – for any neighborhood of u^* it is possible to select such n, that $u^{(n)}$ and further members of the sequence will lie in selected neighborhood of u^* independently of $u^{(0)}$ selection.

Really, for M_{EFR} and any initial point $P_{var}^{(0)}$ we can simply apply the theorem to the set, defined by condition $\Phi(P_{var}) \leq \Phi\big(P_{var}^{(0)}\big)$.

Prove of theorem 1.

Let $u^{(0)} \neq u^*$, then sequence $\Phi\big(u^{(i)}\big)$ decreases monotonically and bounded from below $\Phi(u^*)$ and thus converges to its exact lower bound $c_{inf} = inf\left(\Phi\big(u^{(i)}\big)\right)$. Suppose that $u^{(i)}$ does not converges to u^*. Then there exists such open neighborhood ε of u^*, such that infinite number of $u^{(i)}$ members are outside ε– i.e. $\in K\backslash\varepsilon$. This means that it is possible to choose sub-sequence from $u^{(i)}$ which lies in $K\backslash\varepsilon$. As $K\backslash\varepsilon$ is also compact, according the Bolzano-Weierstrass theorem, we can choose sub-sequence $w^{(i)}$ from this sub-sequence (which also will be sub-sequence of $u^{(i)}$), converging to some not coinciding with u^* point $w^* \in K\backslash\varepsilon$. As Φ is continuous, $\Phi(w^{(i)}) \to \Phi(w^*)$. As $\Phi(w^{(i)})$ is sub-sequence of sequence $\Phi(u^{(i)})$, $\Phi(w^{(i)}) \to c_{inf}$, and $\Phi(w^*) = c_{inf}$. Let's consider sequence $M(w^{(i)})$, which also will be sub-sequence of original sequence $u^{(i)}$. As mapping M is continuous, $M(w^{(i)}) \to M(w^*)$, and $\Phi\left(M(w^{(i)})\right) \to \Phi(M(w^*))$. But as sub-sequence of $\Phi(u^{(i)})$, $\Phi\left(M(w^{(i)})\right) \to c_{inf}$. So $\Phi(M(w^*)) = c_{inf} = \Phi(w^*)$, which contradicts theorem condition $\Phi(M(w^*)) < \Phi(w^*)$. This proves that $u^{(i)}$ converges to u^*.

Let's now prove that convergence is uniform. Let ε – some open neighborhood of point u^*. On compact $K\backslash\varepsilon$ Φ has some minimum value $C_\varepsilon > \Phi(u^*)$. Then set Ω_ε of all points $u \in K$ such that $\Phi(u) < C_\varepsilon$, is also some open neighborhood of u^*, $\Omega_\varepsilon \subseteq \varepsilon$ and (according theorem conditions) $M(\Omega_\varepsilon) \subseteq \Omega_\varepsilon$. Let's define the following sequence of open sets: $\Omega_\varepsilon^{(0)} = \Omega_\varepsilon$, $\Omega_\varepsilon^{(i+1)} = M^{-1}(\Omega_\varepsilon^{(i)})$. As $u^{(i)}$ converges to u^* for any $u^{(0)}$, $K = \bigcup_{i=0}^{\infty} \Omega_\varepsilon^{(i)}$. So we have open cover of compact K – and must contain some finite sub-cover. Maximal number of $\Omega_\varepsilon^{(i)}$ in this sub-cover give required number n.

3. Modified EFR method and its convergence

As can be seen from previous section, the condition that functions $f_i(x_i) - f_i(0)$ should be odd, is very significant for the prove of EFR method convergence, and cannot be dropped. For edges losses on which depends on flow direction, left-hand and right-hand derivatives of $f_i(x_i)$ can be different (if edge properties in laminar flow pattern also depend on flow direction) – and as a result the mapping M_{EFR} cannot be defined as continuous everywhere. Also for such edges equation (15) will not be valid, and inequality (11) may be not valid in some regions.

In practice this leads to oscillation of flow rate sign on some edges in process of EFR iterations. Though this problem usually can be easily solved by usual tricks (for example averaging of oscillating iterations), it would be good to have theoretical justification why and how EFR can be applied for networks with "non-symmetrical" edges. Is it possible to slightly modify EFR method in such a way that it still converges from any initial point for non-symmetrical edges? It found out that it is possible!

Modified EFR method (MEFR) proposed below works for hydraulic networks with the following conditions on edge loss functions:

1) $f_i \in C[-\infty, +\infty]$, i.e. defined and continuous on $\mathbb{R}$;
2) f_i strictly monotonically increase;
3) $f_i \to -\infty$ for $x_i \to -\infty$ and $f_i \to +\infty$ for $x_i \to +\infty$;
4) Right-hand and left-hand derivatives of f_i at $x_i = 0$ exist and are positive: $f'_{i+}(0) > 0$, $f'_{i-}(0) > 0$;
5) Functions $a_{MEFRii+}(x_i) = [f_i(x_i) - f_i(0)]/x_i$ monotonically increase (maybe not strictly) for $x_i > 0$;
6) Functions $a_{MEFRii-}(x_i) = [f_i(x_i) - f_i(0)]/x_i$ monotonically decrease (maybe not strictly) for $x_i < 0$;
7) Functions $a_{MEFRii+}(x_i)$ and $I_{MEFRii+}(x_i) = \int_0^{x_i} [f_i(0) + a_{MEFRii+}(x_i)v - f_i(v)]\, dv$ $\to +\infty$ for $x_i \to +\infty$;

8) Functions $a_{MEFRii-}(x_i)$ and $I_{MEFRii-}(x_i) = \int_0^{x_i}[f_i(0) + a_{MEFRii-}(x_i)v - f_i(v)]\,dv$ $\to +\infty$ for $x_i \to -\infty$

It was already mentioned that conditions 1-3 are standard for existence and unique of CFDP solution. Condition 4 means that edge resistance in laminar flow is taken into account for both directions of the flow. Conditions 5 and 6 are weak convex conditions for edge loss functions and are usually valid in practice. Finally, conditions 7 and 8 means super-linear increase of losses vs flow rate – which is also valid in practice for hydraulic networks.

Note that functions $I_{MEFRii+}(x_i)$ and $I_{MEFRii-}(x_i)$ (which has physical meaning of the area between curve f_i and chord – i.e. difference in energy between real and linearized edge curve) are continuous, monotonic (maybe not strictly), →0 at $x_i \to 0$, and surjectively map correspondingly $(0, +\infty)$ and $(0, -\infty)$ onto $(0, +\infty)$. Let's define functions $a_{MEFRii+}(x_i)$ and $a_{MEFRii-}(x_i)$ continuously in zero point: $a_{MEFRii+}(0) = f'_{i+}(0)$, $a_{MEFRii-}(0) = f'_{i-}(0)$.

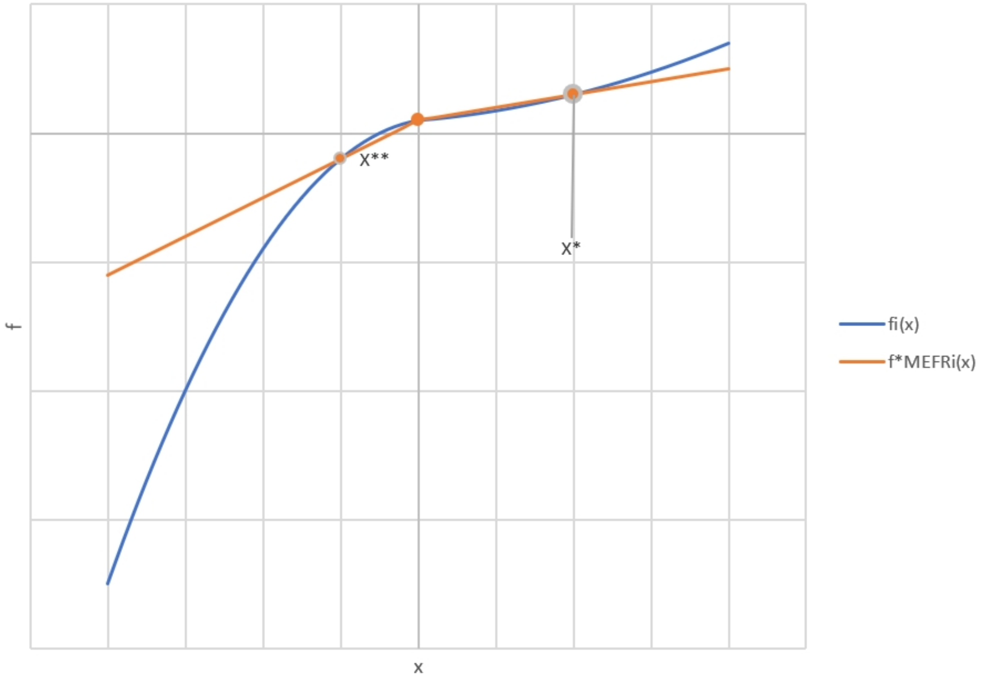

Fig. 3

Modified EFR (MEFR) will replace on each iteration step edge functions $f_i(x_i)$ not with linear ones, but with piecewise linear functions, with different angle on positive and negative semi-axis. Specifically – for P^*_{var} we will define piecewise linear function $f^*_{MEFRi}(x_i)$ as (Fig.3):

On positive semi-axis $x_i \geq 0$

$$\text{At } x_i^* = 0 \; f^*_{MEFRi}(x_i) = f_i(0) + a_{MEFRii+}(0)x_i \tag{17}$$

$$\text{At } x_i^* > 0 \; f^*_{MEFRi}(x_i) = f_i(0) + a_{MEFRii+}(x_i^*)x_i \tag{18}$$

$$\text{At } x_i^* < 0 \; f^*_{MEFRi}(x_i) = f_i(0) + a_{MEFRii+}(x_i^{**})x_i \tag{19}$$

On negative semi-axis $x_i \leq 0$

$$\text{At } x_i^* = 0 \; f^*_{MEFRi}(x_i) = f_i(0) + a_{MEFRii-}(0)x_i \tag{20}$$

$$\text{At } x_i^* > 0 \; f^*_{MEFRi}(x_i) = f_i(0) + a_{MEFRii-}(x_i^{**})x_i \tag{21}$$

$$\text{At } x_i^* < 0 \; f^*_{MEFRi}(x_i) = f_i(0) + a_{MEFRii-}(x_i^*)x_i \tag{22}$$

Here $a_{MEFRii+}(x_i^{**})$ and $a_{MEFRii-}(x_i^{**})$ in (19) and (21) are found from conditions of equality of squares:

$$I_{MEFRii+}(x_i^{**}) = I_{MEFRii-}(x_i^{*}) \tag{23}$$

$$I_{MEFRii-}(x_i^{**}) = I_{MEFRii+}(x_i^{*}) \tag{24}$$

Note that though x_i^{**} can be not unique (when $a_{MEFRii\pm}(x_i)$ and $I_{MEFRii\pm}(x_i)$ are not strictly monotonic), $a_{MEFRii+}(x_i^{**})$ are $a_{MEFRii-}(x_i^{**})$ are correctly define and continuous via x_i^*.

To find x_i^{**} for x_i^* from (23), (24), numerical methods can be used, but for simple functions $f_i(x_i)$ this can be done analytically. For example, for often used «2K method», when $f_i(x_i) = f_i(0) + K_{1+}x_i + K_{2+}x_i|x_i|$ at $x_i \geq 0$ and $f_i(x_i) = f_i(0) + K_{1-}x_i + K_{2-}x_i|x_i|$ at $x_i \leq 0$, integration in (23), (24) will give $x_i^{**} = -x_i^*(K_{2+}/K_{2-})^{1/3}$ at $x_i^* > 0$ and $x_i^{**} = -x_i^*(K_{2-}/K_{2+})^{1/3}$ at $x_i^* < 0$.

The mapping $M_{MEFR}:\mathbb{R}^{N_Q} \to \mathbb{R}^{N_Q}$, which maps any node pressure vector P_{var}^* to node pressure vector in nodes V_Q, which is CFDP solution of system (1), (2), where edge functions are replaced by piecewise linear as described above (such solution exists and unique according [23]) is continuous and has the only fixed point, which correspond to solution of original CFDP. We can always calculate $M_{MEFR}(P_{var}^*)$ by solving 2^{N_E} linear systems for all combinations of different pieces of edge curves and selecting the only solution with correct signs of flow rates. In practice, of course, this usually is not necessary – it is usually enough to solve initially the linear system with pieces correspondent to flow rate signs for P_{var}^* (i.e. to calculate $M_{EFR}(P_{var}^*)$), and then involve other pieces only for edges, where flow rate changes sign.

Similar to section 2 it can be shown that (due to selection of x_i^{**}) $\Phi_{MEFR}^{*'}$ always completely majorizes Φ, so for any point except CFDP solution $\Phi\big(M_{MEFR}(P_{var})\big) < \Phi(P_{var})$. Then according theorem 1 MEFR method converges to CFDP solution for any initial point.

As for those points for which M_{MEFR} mapping does not change sign for edge flow rates, it coincides with M_{EFR}, we also get that for "non-symmetrical" edge network, when EFR iteration stop change flow rate signs, they converge to CFDP solution.

Finally, if CFDP solution has all edge flow rates non-zero, it is always possible to find such $C > min\,\Phi$, that all points with $\Phi \leq C$ has the same signs of flow rates as CFDP solution. Then for any such point as initial point all iteration of EFR method will coincide with MEFR iterations and will converges to CFDP solution.

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